

Answers

J10 pps 120-122 Chapter Exercises

■ p. 120

■ Exercise A.

$$[1.1] \quad x = -1, x = 0$$

$$[1.2] \quad x = 5, x = \frac{1}{2}(1 - i\sqrt{23}), x = \frac{1}{2}(1 + i\sqrt{23})$$

$$[1.3] \quad x = -2, x = 3, x = \frac{1}{2}(1 - i\sqrt{7}), x = \frac{1}{2}(1 + i\sqrt{7})$$

$$[2.1] \quad x = 5, y = 3, z = 1$$

$$[2.2] \quad x = 1, y = -1, x = 5, y \rightarrow 1$$

$$[3.1] \quad \left(a = \frac{3}{2} - \frac{\sqrt{5}}{2}, b = \frac{1}{2}(3 + \sqrt{5})\right) \vee \left(a = \frac{1}{2}(3 + \sqrt{5}), b = \frac{1}{2}(3 - \sqrt{5})\right)$$

$$[3.2] \quad (a = 0, b = -2) \vee (a = 3, b = 1)$$

[4] Suppose α, β are two solutions of $ax^2 + bx + c = 0$. Then $2a\alpha = b^2 \pm \sqrt{b^2 - 4ac}$ and $2a\beta = b^2 \pm \sqrt{b^2 - 4ac}$. In each case, the discriminant $D = b^2 - 4ac$. Since α, β are roots, $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$. Equivalently, $a^2(\alpha + \beta)^2 = b^2$ and $4a^2\alpha\beta = 4ac$. So that $D = a^2(\alpha + \beta)^2 - 4a^2\alpha\beta = a^2((\alpha + \beta)^2 - 4\alpha\beta) = a^2(\alpha - \beta)^2 \square$

[5] Suppose $f(x) = 4x^3 + ax^2 + bx - 25$ is divisible by $2x^2 - 3x - 5$. Then, $f(x)$ is divisible by the factors of $2x^2 - 3x - 5$ which are $(x + 1)(2x - 5)$. By the Factor Theorem, $f(-1) = 0$, $f\left(\frac{5}{2}\right) = 0$.

But, $f(-1) = a - b - 29 = 0$ and $f\left(\frac{5}{2}\right) = \frac{25a}{4} + \frac{5b}{2} + \frac{75}{2} = 0$. Solving for a and b , $a = 4$, $b = -25$.

[6.1] Let $\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = k$ and suppose $a + b + c = 0$. Then,

$$\begin{pmatrix} \frac{b+c}{a} = k \\ \frac{c+a}{b} = k \\ \frac{a+b}{c} = k \end{pmatrix} \iff \begin{pmatrix} b+c = ka \\ c+a = kb \\ a+b = kc \end{pmatrix} \iff \begin{pmatrix} b+c-a-c = ka-kb \\ b+c-a-b = ka-kc \end{pmatrix}$$

$$\iff \begin{pmatrix} b-a = k(a-b) \\ c-a = k(a-c) \end{pmatrix}.$$

Now, if both $a - b = 0$ and $a - c = 0$, then any k , including $k = -1$, makes both equations true.

Supposing that not both $a - b = 0$ and $a - c = 0$, we divide by whichever difference is not zero. That is, if $a - b \neq 0$, then $b - a = k(a - b) \implies \frac{b-a}{a-b} = k \implies -1 = k$.

[6.2] Let $\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = k$ and suppose $a + b + c \neq 0$. Then,

$$\left(\begin{array}{l} \frac{b+c}{a} = k \\ \frac{c+a}{b} = k \\ \frac{a+b}{c} = k \end{array} \right) \iff \left(\begin{array}{l} b+c = ka \\ c+a = kb \\ a+b = kc \end{array} \right)$$

$$\iff (a+b+c) + (a+b+c) = k(a+b+c)$$

$$\iff 2(a+b+c) = k(a+b+c)$$

Since $(a+b+c) \neq 0$, we divide by it obtaining $k = 2$.

[7] Prove that $a + b > 0, ab > 0 \iff a > 0, b > 0$.

\implies

Suppose $a + b > 0$ and $ab > 0$. The WLOG (without loss of generality) $a < b < 0$ or WLOG $0 < a < b$. If $a < b < 0$, then $a + b < b + b < b$. But, since b is negative, $a + b < 0$ which is impossible, since we assume $a + b > 0$. So, $0 < a < b$ which means $a > 0$ and $b > 0$.

\impliedby

Supposing that $a > 0$ and $b > 0$, it is immediate that both the sum and product of a and b are positive.

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[8] Since both the LHS and the RHS are positive, we consider $\text{RHS}^2 - \text{LHS}^2$. That

is,

$$\begin{aligned} & \left(\sqrt{2(a^2 + b^2)} \right)^2 - (|a| + |b|)^2 \\ &= 2(a^2 + b^2) - a^2 - 2|a||b| - b^2 \\ &= a^2 + b^2 - 2|a||b| \\ &\geq 0 \end{aligned}$$

□

[9] Proof (indirect). Suppose that $a^2 + b^2 = c^2$ and that each of a, b , and c is odd. Then

$$a^2 + b^2 = (2m-1)^2 + (2n-1)^2 \text{ for } m \in \mathbb{Z}^+, n \in \mathbb{Z}^+.$$

$$a^2 + b^2 = 4m^2 - 4m + 4n^2 - 4n + 2 = 2(2m^2 - 2m + 2n^2 - 2n + 1) = 2p, p \in \mathbb{Z}, \text{ so } c^2 \text{ is an even}$$

number. Since the square of an odd never an even number, c cannot be odd. This contradicts the supposition that c is odd. Therefore, at least one of a, b, c must be even.

■ **Exercise B.**

[1.1] $x = 1, x = 2, x = 4, x = 5$

[1.2] $x = -3, x = 2, x = \frac{1}{2}(-1 - \sqrt{33}), x = \frac{1}{2}(-1 + \sqrt{33})$

[2.1] \emptyset

[2.2] $(x - \sqrt{2})(x + \sqrt{2})(x^2 + 3)$

[2.3] $(x - \sqrt{2})(x + \sqrt{2})(x - i\sqrt{3})(x + i\sqrt{3})$

[3] Suppose $\alpha + 1, \beta + 1$ are roots of $x^2 - 5x + 8 = 0$. Then

$$(\alpha + 1) + (\beta + 1) = \frac{1}{2}(5 - i\sqrt{7}) + \frac{1}{2}(5 + i\sqrt{7}) \iff \alpha + \beta = 7. \text{ Also,}$$

$$(\alpha + 1)(\beta + 1) = \frac{1}{2}(5 - i\sqrt{7}) \cdot \frac{1}{2}(5 + i\sqrt{7})$$

$$\iff \beta\alpha + \alpha + \beta + 1 = 8$$

$$\iff \alpha\beta + \alpha + \beta = 7$$

$$\implies \alpha\beta = 0.$$

Then consider $x^2 - ax + b = 0$. $\alpha + \beta = 7 \implies \frac{a}{1} = 7 \implies a = 7$. Since $\alpha\beta = 0, b = 0$. $\therefore a = 7, b = 0$.

[4] The roots are $\alpha = \frac{-k - \sqrt{k^2 + 20}}{2}, \beta = \frac{-k + \sqrt{k^2 + 20}}{2}$. So $|\alpha - \beta| = \sqrt{20 + k^2}$. This is minimum when $k^2 = 2\sqrt{5}$.